The limit of an integral

1042. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $f : [0, 1] \to (0, \infty)$ be a continuous function. Calculate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 (1 + x^n)^n f(x)\, dx}.$$

Solution by Arkady Alt, San Jose, CA.

Since f is continuous on $[0, 1]$, it reaches its maximum and minimum values,

$$m = \min_{x\in[0,1]} f(x) \text{ and } M = \max_{x\in[0,1]} f(x),$$

where $m > 0$ since $f([0, 1]) \subset (0, \infty)$. Hence

$$m \int_0^1 (1 + x^n)^n\, dx \le \int_0^1 (1 + x^n)^n f(x)\, dx \le M \int_0^1 (1 + x^n)^n\, dx.$$

Also note that for any $x \in [0, 1]$, $(1 + x^n)^n \le 2^n$ and $1 + x^n \ge 2x^{n/2}$, so that $2^n x^{n^2/2} \le (1 + x^n)^n$. Thus,

$$m \cdot 2^n \int_0^1 x^{n^2/2}\, dx \le m \int_0^1 (1 + x^n)^n\, dx \le \int_0^1 (1 + x^n)^n f(x)\, dx$$
$$\le M \int_0^1 (1 + x^n)^n\, dx \le M \cdot 2^n.$$

We have $\int_0^1 x^{n^2/2}\, dx = \frac{2}{n^2+2}$, $\lim_{n\to\infty} \sqrt[n]{M} = \lim_{n\to\infty} \sqrt[n]{m} = 1$, and, by L'Hopital's rule, $\lim_{n\to\infty} \sqrt[n]{\frac{2}{n^2 + 2}} = 1$. By the squeeze theorem,

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 (1 + x^n)^n f(x)\, dx} = 2.$$

Also solved by Robert A. Agnew, Buffalo Grove, IL and Palm Coast, FL; Byeongkeun Ahn, Sejong Science High S. Seoul, Korea; Michel Bataille, Rouen, France; Hongwei Chen, Christopher Newport U.; Bill Cowieson, Fullerton C.; Bruce E. Davis, St. Louis C.C. Florissant Valley; James Duemmel, Bellingham, WA; Dmitry Fleischman, Santa Monica, CA; Daniel Fritze, Berlin, Germany; Eugene Herman, Grinnell C.; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Cheuk Hang Lee, Massachusetts Inst. Tech.; Luke F. Mannion, St. John's U.; Luis Moreno, SUNY Broome C.C.; Northwestern U. Math Problem Solving Group; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Dipt. Mata., U. Roma Tor Vergata, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Mohammad Riaza-Kermani, Fort Hays State U.; Stephen Scheinberg, Corona del Mar, CA; Joel Schlosberg, Bayside, NY; Haohao Wang and Jerzy Wojdylo (jointly), Southeast Missouri State U.; and the proposer. One incorrect solution was received.